

# THE $G$ -CENTRE AND GRADABLE DERIVED EQUIVALENCES

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**ABSTRACT.** We propose a generalisation for the notion of the centre of an algebra in the setup of graded algebras. Our generalisation, which we call the  $G$ -centre, is designed to control the endomorphism category of the grading shift functors. We show that the  $G$ -centre is preserved by gradable derived equivalences. We discuss links with related notions in superalgebra theory.

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## INTRODUCTION

Consider a finite dimensional algebra  $A$  over a field  $\mathbb{k}$  and the corresponding category  $A\text{-mod}$  of finite dimensional left  $A$ -modules. In this setup, the evaluation of a natural endomorphism of the identity functor on  $A\text{-mod}$  at the left regular  $A$ -module  ${}_A A$  gives rise to the classical isomorphism

$$(0.1) \quad \mathcal{Z}(A) \cong \text{End}_{\mathcal{E}_A}(\text{Id}),$$

between the centre of an algebra and the centre of its module category. In [Ri1, Proposition 9.2], Rickard proved that two derived equivalent algebras have isomorphic centres, providing a fundamental invariant for the study of derived equivalences. When the algebras in question are graded by some group and the derived equivalence suitably preserves this grading, it is easy to show that the centres are isomorphic even as graded algebras. In this paper we take a slightly different view at this situation and introduces a new larger algebra that extends the classical centre of an algebra which we show is preserved by so-called ‘gradable derived equivalences’ between graded algebras.

A motivating example is given by the theory of superalgebras. When associative  $\mathbb{Z}_2$ -graded algebras are interpreted as ‘superalgebras’, there is an alternative notion of the centre, known as *super centre*. Furthermore, in [Go], Gorelik introduced the notion of the *ghost centre* of a superalgebra. This ghost centre is a certain subalgebra containing both the centre and the super centre which turned out to play very important role in studying representations of Lie superalgebras. The natural questions which originated the present study are whether the super centre and the ghost centre could be realised as natural transformations for some endofunctors on the module category and whether these subalgebras are preserved under (certain) derived equivalences.

We start our investigation in a different setting, namely, that of an algebra  $A$  on which some finite group  $H$  acts by automorphisms. This allows us to define the *extended centre*, which is not a subalgebra of  $A$ , but, rather, a subalgebra of  $A \otimes \mathbb{k}H$ . The group action on  $A$  leads to a strict categorical action of  $H$  on

the (derived) module category of  $A$ . We show that the extended centre can be realised as the algebra of natural transformation of the functors which yield this strict categorical action. Furthermore, we prove that derived equivalences which intertwine the actions in a suitable way preserve the extended centres of involved algebras.

When the group  $H$  is abelian, we can interpret it as the Pontryagin dual  $\hat{G}$  of another abelian group  $G$ . The above  $\hat{G}$ -action on  $A$  is then equivalent to a  $G$ -grading. As such, our results above are immediately applicable to graded algebras. However, we develop a different version of the theory, by working with the category of graded modules. The extended centre in this setting is referred to as the  $G$ -centre. We show how the  $G$ -centre can be realised as the algebra of natural transformations of certain functors on the category of graded modules. Then we prove that the  $G$ -centre is preserved under ‘gradable derived equivalences’, as introduced in [CoM].

Then, we return to the special case of  $G = \mathbb{Z}_2$ , thus of that of superalgebras. Our notion of  $G$ -centre is very closely related to the ghost centre. Concretely, it is isomorphic to an exterior direct sum of the super centre and the anti centre, whereas the ghost centre is the sum (not necessarily direct) of the super centre and the anti centre inside the algebra  $A$ . The two notions are thus only different in case some non-zero elements of  $A$  belong to the super and anti centre at the same time, so we can view the  $G$ -centre as a natural lift of the ghost centre. Our general results then yield concrete methods to realise the super centre (and the  $G$ -centre) as endomorphism algebras of certain functors on the supermodule category of a superalgebra. Furthermore, our results show that the super centre and the  $G$ -centre are both preserved under the most canonical definition of derived equivalences between superalgebras. This provides an answer to both our original motivating questions.

The paper is organised as follows. In Section 1 we fix some notation and conventions. In Section 2 we study actions of finite groups on algebras, modules and categories. In Section 3, we obtain our results on the extended centre. In Section 4 we establish some elementary properties of  $G$ -gradings. In Section 5, we obtain our results on the  $G$ -centre. In Section 6, we apply our results to superalgebras and compare with some existing notions in the literature. In Section 7 we point out some natural questions for future research, related to Hochschild cohomology. In Appendix A, we give details on two technical proofs of statements in Section 2 related to strict categorical group actions.

## 1. NOTATION AND CONVENTIONS

We always work over an algebraically closed field  $\mathbb{k}$ . All categories and functors are assumed to be  $\mathbb{k}$ -linear and additive. The category of  $\mathbb{k}$ -linear additive functors on a  $\mathbb{k}$ -linear additive category  $\mathcal{C}$  is denoted by  $\text{Func}(\mathcal{C})$ .

Consider categories  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  and  $\mathcal{D}$ ; functors  $F : \mathcal{A} \rightarrow \mathcal{B}$ ,  $H : \mathcal{C} \rightarrow \mathcal{D}$  and functors  $G_1, G_2 : \mathcal{B} \rightarrow \mathcal{C}$  with a natural transformation  $\eta : G_1 \rightarrow G_2$ . We will use the natural transformation  $H(\eta) : H \circ G_1 \rightarrow H \circ G_2$ , where  $H(\eta)_X := H(\eta_X)$ , for any object  $X$  in  $\mathcal{B}$ . The natural transformation  $\eta_F : G_1 \circ F \rightarrow G_2 \circ F$  is given by  $(\eta_F)_Y := \eta_{F(Y)}$ , for any object  $Y$  in  $\mathcal{A}$ . For an exact functor  $F$  between two abelian categories  $\mathcal{A}$  and  $\mathcal{B}$ , we will use the same notation  $F$  for the corresponding triangulated functor  $\mathcal{D}^b(\mathcal{A}) \rightarrow \mathcal{D}^b(\mathcal{B})$  acting between the corresponding bounded derived categories. We do the same for a natural transformation between two exact functors between abelian categories.

All groups are assumed to be finite. All algebras are over  $\mathbb{k}$  and assumed to be unital, finite dimensional (unless explicitly stated otherwise) and associative. The multiplicative identity of an algebra  $A$  will be denoted by  $1_A$ , or  $1$  if there is no confusion possible. For an algebra  $A$ , we denote the group of algebra automorphisms of  $A$  by  $\text{Aut}(A)$ . We denote

$$\mathcal{E}_A := \text{Func}(A\text{-mod}).$$

For a (finite) group  $H$ , we consider the vector space  $A^{\oplus H}$ , which is just the direct sum of  $|H|$  copies of the vector space  $A$ . However, we keep track of the  $|H|$  copies of  $A$ , by labelling them using elements of  $H$ . The vector space  $A^{\oplus H}$  is then naturally spanned by pairs  $(a, h)$  with  $a \in A$  and  $h \in H$ . General elements can be represented by  $H$ -tuples of elements of  $A$ , viz.  $(x^{(h)} | h \in H) = \sum_{h \in H} (x^{(h)}, h)$  with  $x^{(h)} \in A$ . We will consider two different algebra structures on the underlying vector space  $A^{\oplus H}$ .

For the first algebra structure, we use the notation  $A^{\oplus H}$ , as it is just the direct sum of  $|H|$  copies of the algebra  $A$ . Hence, multiplication on  $A^{\oplus H}$  is given by

$$(a, h)(a', h') = (aa', h)\delta_{h, h'}.$$

Let  $\varepsilon$  be the identity element in  $H$ . The ‘counit’ morphism  $A^{\oplus H} \rightarrow A$  is given by  $(a, h) \mapsto a\delta_{h, \varepsilon}$ .

The second algebra is  $A[H] := A \otimes_{\mathbb{k}} \mathbb{k}H$ , where  $\mathbb{k}H$  is the group algebra of  $H$  over  $\mathbb{k}$ . By definition, multiplication in  $A[H]$  is given by

$$(a, h)(a', h') = (aa', hh').$$

The ‘forgetful’ algebra morphism  $A[H] \rightarrow A$  is given by  $(a, h) \mapsto a$ .

## 2. GROUP ACTIONS

In this section we introduce some notions related to strict categorical actions of groups. Technical proofs of Propositions 4 and 5 are given in Appendix A.

### 2.1. Group actions on algebras and modules.

**2.1.1. Compatible and free actions.** We fix a (finite) group  $H$  with identity element  $\varepsilon$ . An action of  $H$  on an algebra  $A$  is defined to be a group homomorphism  $\phi : H \rightarrow \text{Aut}(A)$ ,  $f \mapsto \phi_f$ . This means that  $\phi_f \circ \phi_h = \phi_{fh}$ , and  $\phi_\varepsilon$  is the identity morphism. Note that we can and will identify  $H$ -actions on  $A$  and  $A^{\text{op}}$ .

Similarly, an action of  $H$  on a vector space  $V$  is defined to be a group homomorphism  $\psi : H \rightarrow \text{Aut}_{\mathbb{k}}(V)$ . If  $V$  is, additionally, an  $A$ -module, then the actions of  $H$  on  $A$  and  $V$  are said to be *compatible* if  $\psi_h(av) = \phi_h(a)\psi_h(v)$ , for all  $h \in H$ ,  $a \in A$  and  $v \in V$ . For any  $\alpha \in \text{Aut}(A)$  and any  $A$ -module  $M$  with underlying vector space  $V$ , we denote by  ${}^\alpha M$  the  $A$ -module with underlying vector space  $V$ , but with the action of  $a \in A$  on  $v \in V$  given by  $\alpha(a) \cdot v$ . The above notion of compatibility is thus equivalent to  $\psi_h \in \text{Hom}_A(M, \phi_h M)$ .

An  $H$ -action  $\kappa$  on  $A$  is *free* if there exists a set of mutually orthogonal idempotents  $\{e_h | h \in H\}$  in  $A$  for which

$$\kappa_h(e_k) = e_{hk}, \text{ for } h, k \in H, \quad \text{and} \quad 1_A = \sum_{h \in H} e_h.$$

For an algebra  $A$  with free  $H$ -action, we define the algebra  $A^H$  as the subalgebra of  $H$ -invariants. Note that  $1_A \in A^H$ .

**Lemma 1.** *Consider an algebra  $A$  with free  $H$ -action  $\kappa$  and the corresponding set  $\{e_h \mid h \in H\}$  of mutually orthogonal idempotents. Then the left ideal  $Ae_\varepsilon$  of  $A$ , when equipped with multiplication  $m : Ae_\varepsilon \times Ae_\varepsilon \rightarrow Ae_\varepsilon$  given by*

$$m(x, y) = \sum_{h \in H} \kappa_h(x)y,$$

*forms an algebra isomorphic to  $A^H$ .*

*Proof.* We have mutually inverse isomorphisms of vector spaces given by

$$Ae_\varepsilon \rightarrow A^H; \quad x \mapsto \sum_{g \in H} \kappa_g(x) \quad \text{and} \quad A^H \rightarrow Ae_\varepsilon; \quad b \mapsto be_\varepsilon.$$

It follows easily that these intertwine the involved multiplications.  $\square$

**2.1.2. The algebras  $A^\phi[H]$  and  $A_\phi[H]$ .** For a group action  $\phi : H \rightarrow \text{Aut}(A)$ , we introduce the algebra  $A^\phi[H]$ , which has  $A^{\oplus H}$  as underlying vector space and is endowed with multiplication

$$(a, h)(b, k) = (\phi_h(b)a, hk).$$

The algebra  $A_\phi[H]$  has  $A^{\oplus H}$  as underlying vector space and is endowed with multiplication

$$(a, h)(b, k) = (a\phi_h(b), hk).$$

Note that we have  $A^\phi[H] \cong (A^{\text{op}})_\phi[H]$ .

## 2.2. Group actions on categories.

**2.2.1. Strict categorical actions.** Let  $\Gamma$  be a strict categorical action of a group  $H$  on a category  $\mathcal{C}$ , *i.e.* we have  $\mathbb{k}$ -linear endofunctors  $\Gamma_h$  on  $\mathcal{C}$ , for each  $h \in H$ , and  $\Gamma_{h_1} \circ \Gamma_{h_2} = \Gamma_{h_1 h_2}$ .

For any object  $X$  in  $\mathcal{C}$ , we introduce the algebra

$$\text{End}_{\mathcal{C}}(\Gamma X) := \bigoplus_{h, k \in H} \text{Hom}_{\mathcal{C}}(\Gamma_h X, \Gamma_k X).$$

This algebra has a free  $H$ -action  $\kappa$ , where  $e_h$  is the identity in  $\text{Hom}_{\mathcal{C}}(\Gamma_h X, \Gamma_h X)$  and  $\kappa_g(\phi) := \Gamma_g(\phi)$ , for any  $\phi \in \text{Hom}_{\mathcal{C}}(\Gamma_h X, \Gamma_k X)$ , extending it to the whole of  $\text{End}_{\mathcal{C}}(\Gamma X)$  by linearity.

In a similar fashion, we can consider the algebra

$$\text{End}_{\text{Func}(\mathcal{C})}(\Gamma) := \bigoplus_{h, k \in H} \text{Hom}_{\text{Func}(\mathcal{C})}(\Gamma_h, \Gamma_k),$$

with  $\kappa_h(\eta)$ , for a natural transformation  $\eta \in \text{Hom}_{\text{Func}(\mathcal{C})}(\Gamma_h, \Gamma_k)$ , given by  $\Gamma_h(\eta)$  and then extended by linearity. Directly from the definitions we have:

**Lemma 2.** *For any object  $X$  in  $\mathcal{C}$ , evaluation yields an algebra morphism*

$$\text{End}_{\text{Func}(\mathcal{C})}(\Gamma) \rightarrow \text{End}_{\mathcal{C}}(\Gamma X); \quad \eta \mapsto \eta_X,$$

*which intertwines the above free  $H$ -actions.*

Consequently, the morphism in the lemma restricts to an algebra morphism

$$(2.1) \quad \text{Ev}_X^\Gamma : \text{End}_{\text{Func}(\mathcal{C})}(\Gamma)^H \rightarrow \text{End}_{\mathcal{C}}(\Gamma X)^H.$$

In some cases we will need a more refined evaluation.

**Definition 3.** The *astute evaluation* is an algebra morphism,

$$\Delta \text{Ev}_X^\Gamma : \text{End}_{\text{Func}(\mathcal{C})}(\Gamma)^H \rightarrow (\text{End}_{\mathcal{C}}(\Gamma X)^H)^{\oplus H},$$

which is given (in the sense of Lemma 1) by

$$\text{Hom}_{\text{Func}(\mathcal{C})}(\text{Id}, \Gamma_g) \ni \eta \mapsto (\Gamma_{k^{-1}}(\eta_{\Gamma_k X}) \mid k \in H).$$

**2.2.2. Intertwining categorical group actions.** Let  $\Gamma$ , resp.  $\Upsilon$ , be strict categorical actions of a group  $H$  on a category  $\mathcal{C}$ , resp.  $\mathcal{D}$ , as in the previous subsection. We say that a  $\mathbb{k}$ -linear functor  $K : \mathcal{C} \rightarrow \mathcal{D}$  *intertwines the actions  $\Gamma$  and  $\Upsilon$*  if we have natural transformations

$$\xi^h : K \circ \Gamma_h \rightarrow \Upsilon_h \circ K, \quad \text{for all } h \in H,$$

where  $\xi^\varepsilon = \text{Id}_K$  and the relation

$$(2.2) \quad \Upsilon_k(\xi^h) \circ \xi_{\Gamma_h}^k = \xi^{kh}$$

is satisfied, for all  $h, k \in H$ . The condition in equation (2.2) is equivalent to saying that the diagram

$$(2.3) \quad \begin{array}{ccc} K \circ \Gamma_{kh} = K \circ \Gamma_k \circ \Gamma_h & \xrightarrow{\xi_{\Gamma_h}^k} & \Upsilon_k \circ K \circ \Gamma_h \\ & \searrow \xi^{kh} & \downarrow \Upsilon_k(\xi^h) \\ & & \Upsilon_{kh} \circ K = \Upsilon_k \circ \Upsilon_h \circ K \end{array}$$

commutes, for all  $h, k \in H$ . The above conditions imply, in particular, that, for any object  $X$  in  $\mathcal{C}$  and any  $h \in H$ , the morphism  $\xi_{\Gamma_{h^{-1}}X}^h$  is invertible, with inverse  $\Upsilon_h(\xi^{h^{-1}})$ . As the functor  $\Gamma_{h^{-1}}$  has inverse  $\Gamma_h$ , this implies that the natural transformation  $\xi^h$  is an isomorphism of functors.

In the particular case where one has the equality  $K \circ \Gamma_h = \Upsilon_h \circ K$ , for all  $h \in H$ , we can take all  $\xi^h$  to be the identity natural transformations and the condition in equation (2.2) is automatically satisfied.

**Proposition 4.** Assume that the functor  $K$  which intertwines the actions  $\Gamma$  and  $\Upsilon$  as above, has a (weak) inverse  $K^{-1}$  given by isomorphisms  $\alpha : K^{-1} \circ K \rightarrow \text{Id}$  and  $\beta : \text{Id} \rightarrow K \circ K^{-1}$  making  $(K, K^{-1})$  a pair of adjoint functors. Then we introduce the natural transformations

$$\eta^h : K^{-1} \circ \Upsilon_h \rightarrow \Gamma_h \circ K^{-1}$$

defined as

$$\eta^h = \alpha_{\Gamma_h \circ K^{-1}} \circ K^{-1}((\xi^h)^{-1})_{K^{-1}} \circ (K^{-1} \circ \Upsilon_h)(\beta).$$

This corresponds to the composition

$$K^{-1} \circ \Upsilon_h \rightarrow K^{-1} \circ \Upsilon_h \circ K \circ K^{-1} \rightarrow K^{-1} \circ K \circ \Gamma_h \circ K^{-1} \rightarrow \Gamma_h \circ K^{-1}.$$

With this definition, the  $\{\eta^h\}$  satisfy the intertwining relations (2.2) for  $K^{-1}$ .

For the proof of Proposition 4, see Appendix A.

When  $\mathcal{C} = \mathcal{D}$  and  $\Gamma = \Upsilon$ , we simply say that  $K$  *commutes with the categorical  $H$ -action  $\Gamma$* .

2.2.3. *Categorical actions and equivalences.* Consider an equivalence  $F : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$  of categories. This induces an equivalence of categories

$$\mathbf{F} : \text{Func}(\mathcal{C}) \xrightarrow{\sim} \text{Func}(\mathcal{D}),$$

where  $\mathbf{F}(K) := F \circ K \circ F^{-1}$ , for a functor  $K$  (an object in  $\text{Func}(\mathcal{C})$ ), and  $\mathbf{F}(\eta) = F(\eta)_{F^{-1}}$ , for a natural transformation  $\eta$  (a morphism in  $\text{Func}(\mathcal{C})$ ). We point out that the equivalence  $\mathbf{F}$  does not necessarily respect composition of functors (it only does it up to isomorphism). In particular, one cannot expect  $\mathbf{F}$  to map a set of functors forming a *strict* group action to a set of functors with the same property. In the following we will continue to refer to objects in  $\text{Func}(\mathcal{C})$  simply as ‘functors’ and morphisms in  $\text{Func}(\mathcal{C})$  as natural transformations.

**Proposition 5.** *Consider an equivalence  $F : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$  which intertwines strict  $H$ -actions  $\Gamma$  on  $\mathcal{C}$  and  $\Upsilon$  on  $\mathcal{D}$ . Then there is an algebra isomorphism*

$$\text{End}_{\text{Func}(\mathcal{C})}(\Gamma) \xrightarrow{\sim} \text{End}_{\text{Func}(\mathcal{D})}(\Upsilon),$$

*which intertwines the  $H$ -action on both algebras as introduced in 2.2.1.*

Note that the algebras in the proposition need not be finite dimensional.

For the proof of Proposition 5, see Appendix A.

**Corollary 6.** *With assumptions as in Proposition 5, we have an algebra isomorphism*

$$\text{End}_{\text{Func}(\mathcal{C})}(\Gamma)^H \cong \text{End}_{\text{Func}(\mathcal{D})}(\Upsilon)^H.$$

Naturally, the analogue of Proposition 5 for evaluations of functors is also true.

**Lemma 7.** *With assumptions as in Proposition 5 and for an object  $X \in \mathcal{C}$ , we have an algebra isomorphism*

$$\text{End}_{\mathcal{C}}(\Gamma X) \cong \text{End}_{\mathcal{D}}(\Upsilon F X),$$

*which intertwines the  $H$ -action on both algebras as introduced in 2.2.1.*

2.2.4. *Category of modules.* A group action  $\phi$  on the algebra  $A$  induces a group action  $\Phi$  on the category  $A\text{-mod}$  as follows. For any  $h \in H$ , let  $\Phi_h$  denote the functor on  $A\text{-mod}$ , which preserves the underlying vector space of modules and preserves morphisms between modules, but twists the  $A$ -action by  $\phi_{h^{-1}} = \phi_h^{-1}$ . This leads to a categorical group action indeed, as, for any  $M \in A\text{-mod}$ , we have

$$\Phi_h \circ \Phi_g(M) = \phi_{h^{-1}} \left( \phi_{g^{-1}} M \right) = \phi_{g^{-1}} \circ \phi_{h^{-1}} M = \Phi_{hg}(M).$$

2.2.5. Consider a category  $\mathcal{C}$  with a strict action  $\Phi$  of  $H$  and an object  $X$  in  $\mathcal{C}$ . We will now formalise the concept of a compatible action on a module of 2.1.1 and use this to define an action on endomorphism algebras.

**Definition 8.** A set of morphisms  $\psi = \{\psi_h, h \in H\}$ , with

$$\psi_h \in \text{Hom}_{\mathcal{C}}(X, \Phi_{h^{-1}} X) \quad \text{and} \quad \Phi_{h^{-1}}(\psi_k) \circ \psi_h = \psi_{kh},$$

and  $\psi_e$  the identity of  $X$ , is called a  $\Phi$ -compatible  $H$ -action on the object  $X$ . If  $X$  admits a  $\Phi$ -compatible  $H$ -action  $\psi$ , the algebra  $\text{End}_{\mathcal{C}}(X)$  admits an  $H$ -action  $\theta = \theta_X^{(\Phi, \psi)}$  given by

$$\theta_g(\alpha) = \psi_{g^{-1}}^{-1} \circ \Phi_g(\alpha) \circ \psi_{g^{-1}},$$

for all  $g \in H$  and  $\alpha \in \text{End}_{\mathcal{C}}(X)$ .

One checks that the above action is well-defined, meaning  $\theta_h \circ \theta_g(\alpha) = \theta_{hg}(\alpha)$  and  $\theta_g(\alpha \circ \beta) = \theta_g(\alpha) \circ \theta_g(\beta)$ .

**Example 9.** Take  $\mathcal{C} = A\text{-mod}$  and  $\Phi$  induced from an  $H$ -action  $\phi : H \rightarrow \text{Aut}(A)$  as in 2.2.4. We can interpret  $\phi_h$  as an element of  $\text{Hom}_A(A, \phi_h A)$ , for each  $h \in H$ . The relation  $\Phi_{h^{-1}}(\phi_k) \circ \phi_h = \phi_{kh}$  follows immediately from the interpretation of both morphisms in  $\text{End}_{\mathbb{k}}(A)$ . Hence, Definition 8 allows to introduce an  $H$ -action  $\theta = \theta_A^{\Phi, \phi}$  on  $\text{End}_A(A) \cong A^{\text{op}}$ . It follows from direct computation that this can be identified with the original  $H$ -action  $\phi$ .

**Lemma 10.** *Under the assumptions of Definition 8, we have an algebra isomorphism*

$$\text{End}_{\mathcal{C}}(\Phi X)^H \xrightarrow{\sim} \text{End}_{\mathcal{C}}(X)_{\theta}[H],$$

where  $\alpha \in \text{Hom}_{\mathcal{C}}(X, \Phi_{h^{-1}} X)$  (as in Lemma 1) is mapped to  $(\Phi_h(\alpha) \circ \psi_{h^{-1}}, h)$ .

*Proof.* We have mutually inverse morphisms of vector spaces given by

$$\text{Hom}_{\mathcal{C}}(X, \Phi_{h^{-1}} X) \rightarrow \text{End}_{\mathcal{C}}(X); \quad \alpha \mapsto \Phi_h(\alpha) \circ \psi_{h^{-1}},$$

and

$$\text{End}_{\mathcal{C}}(X) \rightarrow \text{Hom}_{\mathcal{C}}(X, \Phi_{h^{-1}} X); \quad \alpha \mapsto \Phi_{h^{-1}}(\alpha) \circ \psi_h.$$

Hence, the proposed morphism is an isomorphism of vector spaces. By Lemma 1, for  $\alpha \in \text{Hom}_{\mathcal{C}}(X, \Phi_{h^{-1}} X)$  and  $\beta \in \text{Hom}_{\mathcal{C}}(X, \Phi_{k^{-1}} X)$ , we have  $\alpha\beta = \Phi_{k^{-1}}(\alpha) \circ \beta$ , which is mapped to

$$(\Phi_h(\alpha) \circ \Phi_{hk}(\beta) \circ \psi_{(hk)^{-1}}, hk).$$

On the other hand, by 2.1.2 and Definition 8, the product of  $(\Phi_h(\alpha) \circ \psi_{h^{-1}}, h)$  and  $(\Phi_k(\beta) \circ \psi_{k^{-1}}, k)$  inside  $\text{End}_{\mathcal{C}}(X)_{\theta}[H]$  is given by

$$(\Phi_h(\alpha) \circ \psi_{h^{-1}} \circ \theta_h(\Phi_k(\beta) \circ \psi_{k^{-1}}), hk) = (\Phi_h(\alpha) \circ \psi_{h^{-1}} \circ \psi_{h^{-1}}^{-1} \circ \Phi_{hk}(\beta) \circ \Phi_h(\psi_{k^{-1}}) \circ \psi_{h^{-1}}, hk),$$

and the claim follows.  $\square$

### 3. EXTENDED CENTRE

We fix a (finite) group  $H$  and an algebra  $A$ , for which there is a group homomorphism  $\phi : H \rightarrow \text{Aut}(A)$ ,  $f \mapsto \phi_f$ .

**Definition 11.** The  $\phi$ -extended centre  $\mathcal{Z}^{\phi}(A)$  of  $A$  is the subalgebra of  $A[H]$ , spanned by all  $(a, f)$ , where  $a \in A$  and  $f \in H$ , such that

$$ab = \phi_f(b)a, \quad \text{for all } b \in A.$$

The fact that  $\mathcal{Z}^{\phi}(A)$  is closed under multiplication on  $A[H]$  is immediate. Recalling the definition of the algebras in 2.1.2 leads to the following lemma.

**Lemma 12.**

(i) *The subalgebra  $\zeta^{\phi}(A)$  of  $A^{\phi}[H]$  given by elements  $(a, h)$  satisfying*

$$(a, h)(b, k) = (ab, hk), \quad \text{for all } (b, k) \in A^{\phi}[H],$$

*is isomorphic to  $\mathcal{Z}^{\phi}(A)$ .*

(ii) *The subalgebra  $\zeta_{\phi}(A)$  of  $A_{\phi}[H]$  given by elements  $(a, h)$  satisfying*

$$(a, h)(b, k) = (ba, hk), \quad \text{for all } (b, k) \in A_{\phi}[H],$$

*is isomorphic to  $\mathcal{Z}^{\phi}(A^{\text{op}})$ .*

**3.1. Categorical formulation.** We use the notions introduced in 2.2.1 for the categorical group action  $\Phi$  on  $A\text{-mod}$ , obtained from  $\phi$  as in 2.2.4. The main result of this subsection is the following theorem, which is a generalisation of equation (0.1).

**Theorem 13.** *We have an algebra isomorphism*

$$\mathcal{Z}^\phi(A) \cong \text{End}_{\mathcal{E}_A}(\Phi)^H,$$

*under which  $(a, h) \in \mathcal{Z}^\phi(A)$  is identified with  $\eta : \text{Id} \rightarrow \Phi_{h^{-1}}$ , where  $\eta_M : M \rightarrow \phi_h M$  is given by  $\eta_M(v) = av$ , for all  $v \in M$  and any  $A$ -module  $M$ .*

**Remark 14.** The combination of Theorem 13 and Corollary 6 implies isomorphism of the respectively extended centra of two Morita equivalent algebras with  $H$ -actions for which the induced  $H$ -actions on their module categories are intertwined by the Morita equivalence. We will generalise this statement in Theorem 18.

Now we start the proof of Theorem 13.

**Lemma 15.** *There is an algebra isomorphism*

$$\text{End}_A(\Phi A)^H = \bigoplus_{h \in H} \text{Hom}_A(A, \phi_h A) \rightarrow A^\phi[H],$$

*which maps  $\alpha \in \text{Hom}_A(A, \phi_h A)$  to  $(\alpha(1), h)$  and where the middle expression is as in Lemma 1.*

*Proof.* The proposed morphism is clearly an isomorphism of vector spaces. Now, consider  $\alpha : A \rightarrow \phi_h A$  with  $a := \alpha(1)$  and  $\beta : A \rightarrow \phi_k A$  with  $b := \beta(1)$ . Then  $\alpha\beta = \Phi_{k^{-1}}(\alpha) \circ \beta : A \rightarrow \phi_{hk} A$ , so we have  $\alpha\beta(1) = \phi_h(b)a$ . Hence  $\alpha\beta$  gets mapped to  $(\phi_h(b)a, hk)$ , meaning that we obtain indeed an algebra isomorphism.  $\square$

**Lemma 16.** *For each element  $(a, h) \in \mathcal{Z}^\phi(A)$ , there exists a natural transformation  $\eta : \text{Id} \rightarrow \Phi_{h^{-1}}$  such that  $\eta_M : M \rightarrow \phi_h M$  is given by  $\eta_M(v) = av$ , for all  $v \in M$  and any  $A$ -module  $M$ .*

*Proof.* That  $\eta_M$  is  $A$ -linear follows from definition of  $\mathcal{Z}^\phi(A)$ . For a morphism  $\alpha : M \rightarrow N$ , we have  $\eta_N \circ \alpha = \Phi_{h^{-1}}(\alpha) \circ \eta_M$ , which follows immediately from the fact that  $\Phi_{h^{-1}}(\alpha) = \alpha$  as morphisms of  $\mathbb{k}$ -vector spaces. Thus the family  $\{\eta_M\}$  yields indeed a natural transformation.  $\square$

Now we study the evaluation (2.1) for the left regular  $A$ -module. Evaluations is then automatically injective since  $A$  is a projective generator.

**Lemma 17.** *Denote the composition of the map  $\text{Ev}_A^\Phi : \text{End}_{\mathcal{E}_A}(\Phi)^H \hookrightarrow \text{End}_A(\Phi A)^H$  with the isomorphism in Lemma 15 by*

$$\overline{\text{Ev}}_A^\Phi : \text{End}_{\mathcal{E}_A}(\Phi)^H \hookrightarrow A^\phi[H].$$

*Then the image of  $\overline{\text{Ev}}_A^\Phi$  coincides with the subalgebra  $\zeta^\phi(A) \subset A^\phi[H]$  in Lemma 12(i).*

*Proof.* Consider a natural transformation  $\eta : \text{Id} \rightarrow \Phi_{h^{-1}}$ . Evaluation of  $\eta$  yields a morphism  $\eta_A : A \rightarrow \phi_h A$ , which fits into a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \Phi_{h^{-1}} A \\ \beta \downarrow & & \downarrow \Phi_{h^{-1}}(\beta) \\ A & \xrightarrow{\eta_A} & \Phi_{h^{-1}} A, \end{array}$$



for any morphism  $\beta \in \text{End}_A(A) \cong A^{\text{op}}$ . We take an arbitrary  $b \in A$  and the corresponding  $\beta \in \text{End}_A(A)$  such that  $\beta(1) = b$ . The condition that the above diagram commutes is then equivalent to  $\eta_A(1)b = \phi_h(b)\eta_A(1)$ . We set  $a := \eta_A(1) \in A$  and thus find that  $\text{Im}(\overline{\text{Ev}}_A^\Phi)$  corresponds to those  $(a, h) \in A^\phi[H]$  for which we have  $ab = \phi_h(b)a$ , for all  $b \in A$ . The definition of  $A^\phi[H]$  in 2.1.2 implies that we can characterise these elements  $(a, h)$  equivalently by the condition

$$(a, h)(b, k) = (ab, hk),$$

for all  $(b, k) \in A^\phi[H]$ .  $\square$

*Proof of Theorem 13.* The proposed isomorphism is induced by Lemma 12(i) and  $\overline{\text{Ev}}_A^\Phi$  in Lemma 17. The stated properties of the isomorphism follow by definition of  $\overline{\text{Ev}}_A^\Phi$ .  $\square$

**3.2. Derived equivalences.** The main result of this subsection is the following theorem, which can be viewed as a generalisation of [Ri1, Proposition 9.2]. We denote by  $\mathcal{D}^b(A)$  the bounded derived category of the abelian category  $A\text{-mod}$ .

**Theorem 18.** *Let  $A, B$  be algebras equipped with  $H$ -actions  $\phi : H \rightarrow \text{Aut}(A)$  and  $\omega : H \rightarrow \text{Aut}(B)$ , respectively. Let*

$$F : \mathcal{D}^b(A) \xrightarrow{\sim} \mathcal{D}^b(B)$$

*be an equivalence of triangulated categories such that  $F$  intertwines  $\Phi$  and  $\Omega$  (the categorical actions on  $\mathcal{D}^b(A)$  and  $\mathcal{D}^b(B)$  corresponding to  $\phi$  and  $\omega$ ) as in 2.2.2. Then the algebras  $\mathcal{Z}^\phi(A)$  and  $\mathcal{Z}^\omega(B)$  are isomorphic.*

*Proof.* Let  $\xi^h : F \circ \Phi_h \rightarrow \Omega_h \circ F$  be the natural transformations which give the intertwining relations. Let  $T_\bullet \in \mathcal{D}^b(B)$  be the complex  $F(AA)$ . For each  $h \in H$ , we define

$$\psi_h := \xi_A^{h^{-1}} \circ F(\phi_h) \in \text{Hom}_{\mathcal{D}^b(B)}(T_\bullet, \Omega_{h^{-1}}T_\bullet),$$

where we interpret  $\phi_h$  as an element of  $\text{Hom}_A(A, \Phi_{h^{-1}}A)$ . We calculate, using the definition of  $\xi^{k^{-1}}$  and equation (2.2),

$$\begin{aligned} \Omega_{k^{-1}}(\psi_h) \circ \psi_k &= \Omega_{k^{-1}}(\xi_A^{h^{-1}}) \circ (\Omega_{k^{-1}} \circ F)(\phi_h) \circ \xi_A^{k^{-1}} \circ F(\phi_k) \\ &= \Omega_{k^{-1}}(\xi_A^{h^{-1}}) \circ \xi_{\phi_h A}^{k^{-1}} \circ (F \circ \Phi_{k^{-1}})(\phi_h) \circ F(\phi_k) \\ &= \xi_A^{(hk)^{-1}} \circ F((\Phi_{k^{-1}})(\phi_h) \circ \phi_k) = \xi_A^{(hk)^{-1}} \circ F(\phi_{hk}) = \psi_{hk}. \end{aligned}$$

Hence,  $\psi$  yields an  $\Omega$ -compatible  $H$ -action on  $T_\bullet$  and we can apply Definition 8 to define an action  $\theta = \theta_{T_\bullet}^{\Omega, \psi} : H \rightarrow \Lambda := \text{End}_{\mathcal{D}^b(B)}(T_\bullet)$ . We claim that, under the algebra isomorphism  $A^{\text{op}} \rightarrow \Lambda$  induced by  $A^{\text{op}} \cong \text{End}_A(A)$  and  $F$ , the action  $\theta$  corresponds to the action  $\phi$ . To prove this, we consider  $\alpha \in \text{End}_A(A)$  and calculate

$$\begin{aligned} \theta_h(F(\alpha)) &= (\psi_{h^{-1}})^{-1} \circ (\Omega_h \circ F)(\alpha) \circ \psi_{h^{-1}} \\ &= (\xi_A^h \circ F(\phi_{h^{-1}}))^{-1} \circ (\Omega_h \circ F)(\alpha) \circ \xi_A^h \circ F(\phi_{h^{-1}}) \\ &= (\xi_A^h \circ F(\phi_{h^{-1}}))^{-1} \circ \xi_A^h \circ (F \circ \Phi_h)(\alpha) \circ F(\phi_{h^{-1}}) \\ &= F((\phi_{h^{-1}})^{-1} \circ \Phi_h(\alpha) \circ \phi_{h^{-1}}). \end{aligned}$$

The claim then indeed follows from Example 9. This means, in particular, that  $\Lambda_\theta[H] \cong A_\phi^{\text{op}}[H]$ .

Together with Proposition 23 in Subsection 3.3 this implies that we have an injection

$$\mathcal{Z}^\omega(B) \hookrightarrow \zeta_\theta(\Lambda) \cong \zeta_\phi(A^{\text{op}}).$$

By Lemma 12(ii), we thus have  $\mathcal{Z}^\omega(B) \hookrightarrow \mathcal{Z}^\phi(A)$ . The corresponding reasoning for  $F^{-1}$ , using Proposition 4, gives an inclusion in the other direction, concluding the proof.  $\square$

**3.3. Evaluation.** In this subsection we let  $X_\bullet$  be an arbitrary object in  $\mathcal{D}^b(A)$  and fix a group action  $\phi$  of  $H$  on  $A$ . We assume that  $X_\bullet$  admits a  $\Phi$ -compatible  $H$ -action  $\psi$ . This means that we can apply Definition 8 to construct an  $H$ -action  $\theta$  on  $\text{End}_{\mathcal{D}^b(A)}(X_\bullet)$ .

**Definition 19.** With  $\Lambda := \text{End}_{\mathcal{D}^b(A)}(X_\bullet)$ , we let

$$\zeta_{X_\bullet} : \mathcal{Z}^\phi(A) \rightarrow \Lambda_\theta[H]$$

denote the composition

$$\mathcal{Z}^\phi(A) \xrightarrow{\sim} \text{End}_{\mathcal{E}_A}(\Phi)^H \hookrightarrow \text{End}_{\text{Func}(\mathcal{D}^b(A))}(\Phi)^H \rightarrow \text{End}_{\mathcal{D}^b(A)}(\Phi X_\bullet)^H \xrightarrow{\sim} \Lambda_\theta[H].$$

The first isomorphism is Theorem 13, the second morphism corresponds to the interpretation of natural transformations between exact functors as natural transformations in the derived category, the third morphism is  $\text{Ev}_{X_\bullet}^\Phi$  as in equation (2.1), and the last isomorphism is given by Lemma 10.

**Lemma 20.** *The image of  $\zeta_{X_\bullet}$  is contained in  $\zeta_\theta(\Lambda)$ , with  $\zeta_\theta(\Lambda) \subset \Lambda_\theta[H]$  introduced in Lemma 12(ii).*

*Proof.* We prove the more general statement that the image of the composition

$$\mu : \text{End}_{\text{Func}(\mathcal{D}^b(A))}(\Phi)^H \rightarrow \text{End}_{\mathcal{D}^b(A)}(\Phi X_\bullet)^H \xrightarrow{\sim} \Lambda_\theta[H]$$

is contained in  $\zeta_\theta(\Lambda)$ . We start from a natural transformation  $\eta : \text{Id} \rightarrow \Phi_{h^{-1}}$ , with the corresponding natural transformation  $\Phi_h(\eta) : \Phi_h \rightarrow \text{Id}$ . For any morphism  $\beta \in \text{End}_{\mathcal{D}^b(A)}(X_\bullet)$ , we thus have

$$\beta \circ \Phi_h(\eta_{X_\bullet}) = \Phi_h(\eta_{X_\bullet}) \circ \Phi_h(\beta),$$

which implies

$$\beta \circ \Phi_h(\eta_{X_\bullet}) \circ \psi_{h^{-1}} = \Phi_h(\eta_{X_\bullet}) \circ \phi_{h^{-1}} \circ \phi_{h^{-1}}^{-1} \circ \Phi_h(\beta) \circ \psi_{h^{-1}} = \Phi_h(\eta_{X_\bullet}) \circ \phi_{h^{-1}} \circ \theta_h(\beta).$$

As, by Lemma 10, we have  $\mu(\eta) = (\Phi_h(\eta_{X_\bullet}) \circ \psi_{h^{-1}}, h)$ , the above implies that the image of  $\mu$  is indeed contained in  $\zeta_\theta(\Lambda)$ .  $\square$

Recall from [Ri1, Section 6] that a *tilting complex*  $T_\bullet$  in  $\mathcal{D}^b(A)$  is an object in  $\mathcal{D}^b(A)$  such that

- $\text{Hom}_{\mathcal{D}^b(A)}(T_\bullet, T_\bullet[j]) = 0$ , for all  $j \neq 0$ ;
- $\text{add}(T_\bullet)$  generates  $\mathcal{D}^b(A)$  as a triangulated category.

**Lemma 21.** *Let  $F$  and  $G$  be exact functors on  $A\text{-mod}$ , with  $\eta \in \text{Hom}_{\mathcal{E}_A}(F, G)$ . If  $\eta_A \neq 0$ , then  $\eta_{T_\bullet} \neq 0$ , for any tilting complex  $T_\bullet$  in  $\mathcal{D}^b(A)$ , with  $\eta$  interpreted as a natural transformation for  $\mathcal{D}^b(A)$ .*

*Proof.* Let  $X$  and  $Y$  be  $A$ - $A$ -bimodules representing  $F$  and  $G$ , respectively, and  $f : X \rightarrow Y$  be a (non-zero) homomorphism of bimodules representing  $\eta$ . Let  $\mathcal{K}^-(A\text{-proj})$  be the homotopy category of complexes of projective  $A$ -modules bounded from the right and  $\mathcal{K}^-(A\text{-proj})_c$  the full subcategory of complexes having only finitely many non-zero homologies. Then the canonical embedding of  $\mathcal{K}^-(A\text{-proj})_c$  into  $\mathcal{D}^b(A)$  is an equivalence of triangulated categories. We can view functors  $F$  and  $G$  as functors from  $\mathcal{K}^-(A\text{-proj})_c$  to  $\mathcal{D}^b(A)$  and then they are given by tensoring with the corresponding bimodules. In this realisation,  $\eta$  corresponds to  $f \otimes \text{id}$ , applied component

wise. In this way we see that  $F$ ,  $G$  and  $\eta$  commute with the cone construction in the sense that, for any  $P_\bullet, Q_\bullet$  in  $\mathcal{K}^b(A)$  and  $\xi : P_\bullet \rightarrow Q_\bullet$ , we have actual equalities

$$\text{Cone}(F(\xi)) = F(\text{Cone}(\xi)); \quad \text{Cone}(G(\xi)) = G(\text{Cone}(\xi)); \quad \text{Cone}(\eta_\xi) = \eta_{\text{Cone}(\xi)}.$$

Now, assume that  $T_\bullet$  is a tilting complex in  $\mathcal{K}^-(A\text{-proj})_c$  and that  $\eta_{T_\bullet} = 0$ . Then the previous paragraph implies that  $\eta_{Q_\bullet} = 0$ , for any  $Q_\bullet$  in the triangulated category generated by  $T_\bullet$ . As the latter triangulated category is the whole of  $\mathcal{D}^b(A)$ , it follows that, in particular,  $\eta_A = 0$ , where  $A$  is the projective generator of  $A\text{-mod}$ , a contradiction.  $\square$

**Lemma 22.** *The morphism  $\zeta_{X_\bullet}$  is injective, if  $X_\bullet$  is a tilting complex in  $\mathcal{D}^b(A)$ .*

*Proof.* By Lemma 21, the composition

$$\text{End}_{\mathcal{E}_A}(\Phi)^H \rightarrow \text{End}_{\text{Func}(\mathcal{D}^b(A))}(\Phi)^H \rightarrow \text{End}_{\mathcal{D}^b(A)}(\Phi T_\bullet)^H$$

is injective. As the other morphisms in the composition for  $\zeta_{T_\bullet}$  in Definition 19 are isomorphisms, the lemma follows.  $\square$

We summarise the results of this subsection in the following proposition.

**Proposition 23.** *For any tilting complex  $T_\bullet$  in  $\mathcal{D}^b(A)$  which admits a  $\Phi$ -compatible  $H$ -action  $\psi$ , we have an injection*

$$\mathcal{Z}^\phi(A) \hookrightarrow \zeta_\theta(\Lambda),$$

with  $\theta$  as in Definition 8 and  $\Lambda := \text{End}_{\mathcal{D}^b(A)}(T_\bullet)$ .

#### 4. GRADINGS

**4.1. The abelian group  $G$ .** We fix a (finite) abelian group  $G$  for the rest of the paper, which is assumed to satisfy the property that  $|G|$  is not divisible by  $\text{char}(\mathbb{k})$ . As  $G$  will be used to define gradings, we adopt the convention to denote its operation by  $+$ , the identity element by  $0_G$  or  $0$  and the inverse of  $g \in G$  by  $-g$ .

Denote by  $\hat{G}$  the character group, or Pontryagin dual, of  $G$ . The group  $\hat{G}$  thus consists of all group homomorphisms  $G \rightarrow \mathbb{k}^\times$ , and the multiplication is point-wise. Note that, under our assumption on  $|G|$  and  $\text{char}(\mathbb{k})$ , and as  $\mathbb{k}$  is algebraically closed, we have the orthogonality relations

$$(4.1) \quad \sum_{\chi \in \hat{G}} \chi(g) \chi(-h) = |G| \delta_{g,h} \quad \text{and} \quad \sum_{g \in G} \chi(g) \chi'(-g) = |G| \delta_{\chi, \chi'},$$

see e.g. [Ap, Sections 6.5-6.7].

Recall the algebras  $A^{\oplus G}$  and  $A[\hat{G}]$  as introduced in Section 1.

**Lemma 24.** *There is an algebra isomorphism  $A[\hat{G}] \xrightarrow{\sim} A^{\oplus G}$  given by*

$$(x, \chi) \mapsto (\chi(h)x \mid h \in G),$$

for all  $x \in A$  and  $\chi \in \hat{G}$ .

*Proof.* The given morphism is an algebra morphism. Moreover, the map

$$A^{\oplus G} \rightarrow A[\hat{G}]; \quad (x^{(g)} \mid g \in G) \mapsto \frac{1}{|G|} \left( \sum_{l \in G} \eta(-l) x^{(l)} \mid \eta \in \hat{G} \right)$$

is an inverse of this morphism, as follows from a direct computation using equations (4.1).  $\square$

The isomorphism in Lemma 24 clearly leads to a commuting triangle of algebra morphisms

$$(4.2) \quad \begin{array}{ccc} A[\hat{G}] & \xrightarrow{\sim} & A^{\oplus G} \\ & \searrow & \swarrow \\ & A, & \end{array}$$

where the two downward arrows are the forgetful and the counit morphisms as introduced in Section 1.

#### 4.2. Gradings.

**4.2.1. Graded vector spaces.** For the group  $G$ , we introduce the category  $\mathbb{k}\text{-gmod}$ . Its objects are finite dimensional  $\mathbb{k}$ -vector spaces  $V$  equipped with a  $G$ -grading,

$$V = \bigoplus_{g \in G} V_g.$$

The morphisms are those respecting the grading, *i.e.* homogeneous  $\mathbb{k}$ -linear maps of degree 0. We also use the notation  $\text{hom}_{\mathbb{k}}$  for  $\text{Hom}_{\mathbb{k}\text{-gmod}}$ , with similar convention for  $\text{End}$ . For any  $G$ -graded  $\mathbb{k}$ -vector space  $V$ , we write  $\partial(v) = g$  for  $v \in V_g$ . Whenever  $\partial$  is used, we assume that the element on which it acts is homogeneous. For two  $G$ -graded vector spaces  $V$  and  $W$ , their direct sum is  $G$ -graded with  $(V \oplus W)_g = V_g \oplus W_g$ .

For any  $g \in G$  and a  $G$ -graded vector space  $V$ , we define the  $G$ -graded vector space  $\Pi_g V$ , which coincides with  $V$  as an ungraded vector space, but with grading given by  $(\Pi_g V)_h = V_{h+g}$ . For any  $v \in V$ , we use the notation  $\Pi_g v$  for the element in  $\Pi_g V$  identified with  $v$  through the equalities  $(\Pi_g V)_h = V_{h+g}$ . In particular,

$$(4.3) \quad v \in V_h \quad \text{implies that} \quad \Pi_g(v) \in (\Pi_g V)_{h-g}.$$

In other words, we have  $\partial(\Pi_g v) = \partial(v) - g$ .

We will interpret  $\Pi_g$  as an endofunctor of  $\mathbb{k}\text{-gmod}$ , defined on a morphism  $f : V \rightarrow W$  as  $\Pi_g(f)(\Pi_g v) = \Pi_g f(v)$ , for any  $v \in V$ . In particular,  $\Pi_0 = \text{Id}$  and  $\Pi_{g_1} \Pi_{g_2} = \Pi_{g_1+g_2}$ , so the functors  $\{\Pi_g \mid g \in G\}$  form a group isomorphic to  $G$  and  $\Pi$  is a strict categorical  $G$ -action on  $\mathbb{k}\text{-gmod}$ , in the sense of 2.2.1.

**4.2.2. Graded algebras.** A  $G$ -graded algebra  $A$  is a  $\mathbb{k}$ -algebra,  $G$ -graded as a vector space, such that  $A_g A_h \subset A_{g+h}$ , for  $g, h \in G$ . It follows immediately that  $1 \in A_0$ . A  $G$ -graded  $A$ -module is a  $G$ -graded  $\mathbb{k}$ -vector space  $V = \bigoplus_{g \in G} V_g$  such that the action of  $A$  satisfies  $A_g V_h \subset V_{h+g}$ . Recall that both  $|G|$  and  $\dim_{\mathbb{k}} A$  are finite. We define the category  $A\text{-gmod}$  as the category of finite dimensional  $G$ -graded  $A$ -modules with morphisms being  $A$ -linear morphisms of  $G$ -graded vector spaces. Note that we can consider  $\mathbb{k}$  as a  $G$ -graded  $\mathbb{k}$ -algebra concentrated in degree zero, then the notation  $\mathbb{k}\text{-gmod}$  is consistent with this interpretation. Morphism spaces in the category  $A\text{-gmod}$  will be denoted by  $\text{hom}_A$ .

We denote the exact functor forgetting the  $G$ -grading by

$$F^g : A\text{-gmod} \rightarrow A\text{-mod}.$$

When non-essential, we will sometimes leave out reference to this forgetful functor. We also identify  $F^g M$  and  $F^g \Pi_g M$ , for a  $G$ -graded module  $M$ .

For an algebra  $B$  with free  $G$ -action  $\kappa$  as in 2.1.1, we define a  $G$ -grading on the algebra of invariants  $B^G$ . For this, we set

$$(4.4) \quad (B^G)_g := e_g B e_0,$$

where the right-hand side is to be interpreted via the isomorphism  $B^G \cong B e_0$  of Lemma 1.

4.2.3. We set

$$\mathcal{E}_A^G := \text{Func}(A\text{-gmod}).$$

For any  $g \in G$ , the functor  $\Pi_g$  of 4.2.1 induces an endofunctor of  $A\text{-gmod}$ , which we also denote by  $\Pi_g \in \mathcal{E}_A^G$ . Clearly  $\Pi$  yields a strict  $G$ -action on  $A\text{-gmod}$  in the sense of 2.2.1.

By 2.2.1, the algebra  $\text{end}_A(\Pi A)$  admits a free  $G$ -action. Hence the corresponding algebra of invariants is  $G$ -graded by convention (4.4).

**Lemma 25.** *We have an isomorphism of  $G$ -graded algebras*

$$\text{end}_A(\Pi A)^G \xrightarrow{\sim} A^{\text{op}},$$

where  $\alpha \in \text{hom}_A(A, \Pi_g A)$ , interpreted as in Lemma 1, is mapped to  $\Pi_g \alpha(1)$ .

*Proof.* For  $\alpha \in \text{hom}_A(A, \Pi_g A)$ , we have  $\alpha(1) = \Pi_g a$ , for some  $a \in A_g$ . The described map is thus an isomorphism of  $G$ -graded vector spaces. Now take  $\alpha \in \text{hom}_A(A, \Pi_g A)$  and  $\beta \in \text{hom}_A(A, \Pi_h A)$ , Lemma 1 implies that  $\alpha\beta = \Pi_h(\alpha) \circ \beta$ . Since we have  $\Pi_h(\alpha) \circ \beta(1) = \Pi_{g+h} \alpha\beta(1)$  with  $a = \Pi_g \alpha(1)$  and  $b = \Pi_h \beta(1)$ , this concludes the proof.  $\square$

More generally, we have the following result, which is proved similarly. Set  $\mathcal{D}^g := \mathcal{D}^b(A\text{-gmod})$  and  $\mathcal{D} := \mathcal{D}^b(A\text{-mod})$ .

**Lemma 26.** *For any  $Y_\bullet \in \mathcal{D}^b(A\text{-gmod})$ , with  $\Lambda := \text{End}_{\mathcal{D}}(F^g Y_\bullet)$ , the forgetful functor  $F^g$  induces an algebra isomorphism*

$$\text{End}_{\mathcal{D}^g}(\Pi Y_\bullet)^G \xrightarrow{\sim} \Lambda.$$

This lemma thus allows us to equip any endomorphism algebra  $\Lambda$  of a gradable object  $Y_\bullet$  in  $\mathcal{D}$  with a  $G$ -grading, where

$$(4.5) \quad \Lambda_g \cong \text{Hom}_{\mathcal{D}^g}(Y_\bullet, \Pi_g Y_\bullet).$$

4.2.4. *Conventions for gradings.* We introduce some canonical gradings on types of algebras, which will be assumed throughout the paper.

- (A) For any algebra  $A$  and abelian group  $H$ , the algebra  $A[H]$  is  $H$ -graded, with  $A[H]_h = \{(a, h) \mid a \in A\}$ .
- (B) If  $A$  is  $G$ -graded, then the vector space  $A^{\oplus |H|}$  is  $G$ -graded as a direct sum of graded vector spaces. Both  $A[H]$  and  $A^{\oplus H}$  are  $G$ -graded algebras for this grading.

With convention (B), the algebra isomorphism of Lemma 24 is, in fact, a  $G$ -graded algebra isomorphism.

**4.3. Actions versus gradings.** For a vector space  $V$ , or an algebra  $A$ , we have a natural correspondence between  $G$ -gradings and  $\hat{G}$ -actions. The theory of graded algebras is thus equivalent to the theory of commutative group actions on algebras, when  $|G|$  is not divisible by  $\text{char}(\mathbb{k})$ .

**Lemma 27.** *Let  $V$  be a finite dimensional  $\mathbb{k}$ -vector space.*

- (i) *If  $V$  is  $G$ -graded, define  $\psi_\chi \in \text{End}_{\mathbb{k}}(V)$  by  $\psi_\chi(v) = \chi(\partial v)v$ . Then  $\psi$  is a  $\hat{G}$ -action on  $V$ .*
- (ii) *If  $V$  has a  $\hat{G}$ -action  $\psi$ , we define*

$$V_g := \{v \in V \mid \psi_\chi(v) = \chi(g)v, \text{ for all } \chi \in \hat{G}\}.$$

*Then  $V = \bigoplus_{g \in G} V_g$  is a  $G$ -graded vector space.*

*The two procedures are each others' inverses.*

*If  $V$  has an algebra structure, the above definitions interchange group actions and gradings for the algebra structure. For an algebra  $A$  with module  $V$ , the above definitions interchange compatible group actions and  $G$ -graded module structures.*

*Proof.* Statements (i) and (ii) are immediate consequences of the orthogonality relations (4.1). The other claims follow from direct computations.  $\square$

For  $V$  as in Lemma 27(i), we simply write  $v_\chi$ , for  $\psi_\chi(v) = \chi(\partial v)v$ . The lemma thus implies, in particular, that, for a  $G$ -graded algebra  $A$ , we have a group morphism

$$(4.6) \quad \phi : \hat{G} \rightarrow \text{Aut}(A), \quad \phi_\chi(a) = a_\chi, \quad \text{for all } \chi \in \hat{G} \text{ and } a \in A.$$

## 5. THE $G$ -CENTRE

Fix a finite dimensional unital associative  $G$ -graded  $\mathbb{k}$ -algebra  $A$ . We apply Definition 11 to the  $\hat{G}$ -action  $\phi$  in equation (4.6). Consider the algebra  $A[\hat{G}]$  with the  $G$ -grading of convention (B) and the  $\hat{G}$ -grading of convention (A). The algebra  $\mathcal{Z}^\phi(A)$  will in this case be denoted by  $\mathcal{Z}^G(A)$ .

**Definition 28.**

- (i) The  $G$ -centre  $\mathcal{Z}^G(A)$  is the unital  $G \times \hat{G}$ -graded subalgebra of  $A[\hat{G}]$ , where, for given  $g \in G$  and  $\chi \in \hat{G}$ , the space  $\mathcal{Z}^G(A)_{(g,\chi)}$  is spanned by all  $(x, \chi)$ , for which  $x \in A_g$  and

$$xy = y_\chi x, \quad \text{for all } y \in A.$$

- (ii) The image of  $\mathcal{Z}^G(A)$  under the forgetful morphism  $A[\hat{G}] \twoheadrightarrow A$  is denoted by  $\underline{\mathcal{Z}}^G(A)$ .

The algebra  $\underline{\mathcal{Z}}^G(A)$  is still naturally  $G$ -graded, but will, in general, no longer be  $\hat{G}$ -graded, see Example 43.

**Remark 29.** Most of the multiplication in the algebra  $\mathcal{Z}^G(A)$  is zero. Consider  $g, h \in G$  and  $x \in A_g, y \in A_h$  such that the elements  $(x, \chi), (y, \chi') \in A[\hat{G}]$  belong to  $\mathcal{Z}^G(A)$ . Then clearly

$$(x, \chi)(y, \chi') = 0, \quad \text{unless } \chi'(g)\chi(h) = 1.$$

**5.1. Categorical and alternative formulations.** We can express the  $G$ -centre naturally in a generalisation of (0.1). Contrary to the previous generalisation of (0.1) to  $\mathcal{Z}^G(A)$  in Theorem 13, we use the category  $A\text{-gmod}$  instead of  $A\text{-mod}$ .

**Theorem 30.** *As  $G$ -graded algebras, we have*

$$\mathcal{Z}^G(A)^{\text{op}} \cong \left( \text{End}_{\mathcal{E}_A^G}(\Pi) \right)^G.$$

This theorem will be proved in the next subsection.

The next lemma demonstrates how  $\mathcal{Z}^G(A)$  can be defined as a subalgebra of  $A^{\oplus G}$ , instead of as a subalgebra of  $A[\hat{G}]$ . The proof is a straightforward application of definitions and isomorphisms in (the proof of) Lemma 24.

**Lemma 31.** *Under the isomorphism  $A[\hat{G}] \rightarrow A^{\oplus G}$  of  $G$ -graded algebras given in Lemma 24, the  $G$ -centre  $\mathcal{Z}^G(A)$  is mapped to the subalgebra of  $A^{\oplus G}$  consisting of all  $G$ -tuples  $(x^{(g)} | g \in G)$ , for which*

$$x^{(g)}y = yx^{(g+h)}, \quad \text{for all } y \in A_h.$$

**Corollary 32.** *The algebra  $\underline{\mathcal{Z}}^G(A)$  is a  $G$ -graded subalgebra of  $A$  described equivalently as:*

- (a) *the span of the elements  $x \in A$  for which there exists  $\chi \in \hat{G}$  such that  $xy = y_\chi x$  for all  $y \in A$ ;*
- (b) *the image under the counit morphism  $A^{\oplus G} \rightarrow A$  of  $\mathcal{Z}^G(A) \subset A^{\oplus G}$ .*

*Proof.* Part (a) follows immediately from Definition 28. Part (b) follows from Definition 28(ii) and commutative diagram (4.2).  $\square$

**5.2. Evaluation.** We study the evaluation (2.1)

$$\text{Ev}_M^\Pi : \text{End}_{\mathcal{E}_A^G}(\Pi)^G \rightarrow \text{end}_A(\Pi M)^G,$$

and the astute evaluation of Definition 3,

$$\Delta \text{Ev}_M^\Pi : \text{End}_{\mathcal{E}_A^G}(\Pi)^G \rightarrow (\text{end}_A(\Pi M)^G)^{\oplus G}.$$

First we apply  $\Delta \text{Ev}^\Pi$  to the left regular module. By Lemma 25, we have an isomorphism  $(\text{end}_A(\Pi A)^G)^{\oplus G} \cong (A^{\text{op}})^{\oplus G}$ . We denote by  $\Delta \overline{\text{Ev}}_A^\Pi$  the composition of  $\Delta \text{Ev}_A^\Pi$  with this isomorphism.

**Proposition 33.** *The astute evaluation morphism*

$$\Delta \overline{\text{Ev}}_A^\Pi : \text{End}_{\mathcal{E}_A^G}(\Pi)^G \rightarrow (A^{\oplus G})^{\text{op}}$$

*is injective and has  $(\mathcal{Z}^G(A))^{\text{op}}$  as the image.*

*Proof.* The injectivity of  $\Delta \overline{\text{Ev}}_A^\Pi$  follows from the fact that  $\oplus_k \Pi_k A$  is a projective generator of  $A\text{-gmod}$ . Any multiplication of elements in  $A$  will be interpreted as multiplication inside  $A$ , not  $A^{\text{op}}$ .

Now consider a natural transformation  $\eta : \text{Id} \rightarrow \Pi_g$ . The image under  $\Delta \overline{\text{Ev}}_A^\Pi$  is given by

$$(5.1) \quad (x^{(k)} \mid k \in G) \in (A^{\text{op}})^{\oplus G}, \quad \text{with } x^{(k)} := \Pi_{-g-k}(\eta_{\Pi_k A}(\Pi_k 1)).$$

Consider arbitrary  $h \in G$  and  $a \in A_h$ . This  $a$  defines, for all  $l \in G$ , a morphism  $\alpha_l : \Pi_l A \rightarrow \Pi_{l+h} A$  given by  $\Pi_l b \mapsto \Pi_{l+h} ba$ , for all  $b \in A$ . Note that, by definition,  $\Pi_{l'}(\alpha_l) = \alpha_{l+l'}$ . Since  $\eta$  is a natural transformation, we have a commuting diagram

$$\begin{array}{ccc} \Pi_k A & \xrightarrow{\eta_{\Pi_k A}} & \Pi_g \Pi_k A \\ \downarrow \alpha_k & & \downarrow \alpha_{g+k} \\ \Pi_{h+k} A & \xrightarrow{\eta_{\Pi_{h+k} A}} & \Pi_g \Pi_{h+k} A \end{array}$$

meaning that  $x^{(k)}a = ax^{(h+k)}$ . By Lemma 31, we thus find that  $(x^{(k)} \mid k \in G) \in \mathcal{Z}^G(A)$ . This implies that the image of  $\Delta \overline{\text{Ev}}_A^\Pi$  is contained in  $(\mathcal{Z}^G(A))^{\text{op}} \subset (A^{\oplus G})^{\text{op}}$ .

Now, start from an arbitrary  $(x^{(k)} \mid k \in G) \in \mathcal{Z}^G(A)_g$ , for  $g \in G$ . We want to define  $\eta : \text{Id} \rightarrow \Pi_g$ . For any  $M \in A\text{-gmod}$ , we define a morphism

$$\eta_M : M \rightarrow \Pi_g M, \quad \text{by } v \mapsto \Pi_g x^{(-h)} v, \quad \text{for } v \in M_h.$$

This morphism is  $A$ -linear by the properties of  $(x^{(k)} \mid k \in G) \in \mathcal{Z}^G(A)_g$ .

For any morphism  $\alpha : M \rightarrow N$ , we claim that  $\eta_N \circ \alpha = \Pi_g(\alpha) \circ \eta_M$ . Indeed, for  $v \in M_h$ , we have

$$\eta_N \circ \alpha(v) = \Pi_g x^{(-h)} \alpha(v) = \Pi_g \alpha(x^{(-h)} v) = \Pi_g(\alpha) \left( \Pi_g x^{(-h)} v \right) = \Pi_g(\alpha) \circ \eta_M(v),$$

so  $\eta$  is a natural transformation. Thus we find that the image of  $\Delta \overline{\text{Ev}}_A^\Pi$  is, in fact, equal to  $(\mathcal{Z}^G(A))^{\text{op}}$ , concluding the proof.  $\square$

Proposition 33 implies Theorem 30. Additionally, we also have the following two corollaries.

**Corollary 34.** *Consider the  $G \times \hat{G}$ -grading on  $\mathcal{Z}^G(A)$  as given by Definition 28(i). The algebra isomorphism in Theorem 30 restricts to vector space isomorphisms*

$$\mathcal{Z}^G(A)_{g,\chi} \cong \{ \eta \in \text{Hom}_{\mathcal{E}_A^G}(\text{Id}, \Pi_g) \mid \eta_{\Pi_k} = \chi(k) \Pi_k(\eta), \quad \text{for all } k \in G \}.$$

*Proof.* The  $G$ -grading is immediate. Now, consider some  $\chi \in \hat{G}$  and  $(x, \chi) \in \mathcal{Z}^G(A)_{g,\chi}$  in the realisation as a subalgebra of  $A[\hat{G}]$ . By Lemma 24, this corresponds to

$$(\chi(h)x \mid h \in G) \in \mathcal{Z}^G(A)$$

in the realisation inside  $A^{\oplus G}$ . The corresponding natural transformation  $\eta : \text{Id} \rightarrow \Pi_g$  satisfies

$$\chi(h) \Pi_{g+h} x = \eta_{\Pi_h A}(\Pi_h 1), \quad \text{for all } h \in G.$$

by equation (5.1). In particular, it follows immediately that

$$(\eta_{\Pi_k})_{\Pi_h A} = \chi(k) \Pi_k(\eta)_{\Pi_h A}, \quad \text{for all } k, h \in G.$$

As  $\oplus_h \Pi_h A$  is a projective generator, the claim follows.  $\square$

We compose  $\text{Ev}_A^\Pi$  with the isomorphism in Lemma 25.

**Corollary 35.** *The image of*

$$\overline{\text{Ev}}_A^\Pi : \text{End}_{\mathcal{E}_A^G}(\Pi)^G \rightarrow A^{\text{op}}$$

*is given by  $\underline{\mathcal{Z}}^G(A)^{\text{op}}$ .*



*Proof.* We have the following commuting triangle of algebra morphisms

$$\begin{array}{ccc} \text{End}_{\mathcal{E}_A^G}(\Pi)^G & \xrightarrow{\Delta \overline{\text{Ev}}_A^\Pi} & (A^{\text{op}})^{\oplus G} \\ & \searrow \overline{\text{Ev}}_A^\Pi & \downarrow \\ & & A^{\text{op}} \end{array}$$

in which the vertical arrow is given by the counit morphism. The result hence follows from Proposition 33 and Corollary 32(b).  $\square$

In analogy with Definition 19, we introduce the following composition of morphisms. We set  $\mathcal{D}^g = \mathcal{D}^b(A\text{-gmod})$  and  $\mathcal{D} = \mathcal{D}^b(A\text{-mod})$ .

**Definition 36.** Consider  $X_\bullet \in \mathcal{D}^g$ , with  $\Lambda := \text{End}_{\mathcal{D}}(X_\bullet)$  equipped with the  $G$ -grading inherited in Lemma 26 and equation (4.5). The morphism

$$\Delta\zeta_{X_\bullet} : \mathcal{Z}^G(A)^{\text{op}} \rightarrow \Lambda^{\oplus G}$$

of  $G$ -graded algebras is given by the composition

$$\mathcal{Z}^G(A)^{\text{op}} \xrightarrow{\sim} \text{End}_{\mathcal{E}_A^G}(\Pi)^G \hookrightarrow \text{End}_{\text{Func}(\mathcal{D}^g)}(\Pi)^G \rightarrow (\text{End}_{\mathcal{D}^g}(\Pi X_\bullet))^{\oplus G} \xrightarrow{\sim} \Lambda^{\oplus G}.$$

The first isomorphism is Proposition 33, the third morphism is  $\Delta \text{Ev}_{X_\bullet}^\Pi$  in Definition 3 and the last isomorphism is induced from the one in Lemma 26.

**Lemma 37.** With notation as in Definition 36, the image of  $\Delta\zeta_{X_\bullet}^\Pi$  is contained in  $\mathcal{Z}^G(\Lambda^{\text{op}})^{\text{op}}$ . The corresponding morphism

$$\Delta\zeta_{X_\bullet}^\Pi : \mathcal{Z}^G(A)^{\text{op}} \rightarrow \mathcal{Z}^G(\Lambda^{\text{op}})^{\text{op}}$$

is a morphism of  $G \times \hat{G}$ -graded algebras.

*Proof.* The image under  $\Delta\zeta_{X_\bullet}^\Pi$  of an element in  $\mathcal{Z}^G(A)$  corresponding to the natural transformation  $\eta : \text{Id} \rightarrow \Pi_g$  is given by

$$(\alpha^{(k)} \mid k \in G) \in \Lambda^{\oplus G}, \quad \text{with } \alpha^{(k)} := F^g(\eta_{\Pi_k X_\bullet}).$$

For an arbitrary  $\beta \in \text{Hom}_{\mathcal{D}^g}(X_\bullet, \Pi_h X_\bullet)$ , the fact that  $\eta$  is a natural transformation implies that

$$\Pi_{g+h}(\beta) \circ \eta_{\Pi_k X_\bullet} = \eta_{\Pi_{k+h} X_\bullet} \circ \Pi_k(\beta),$$

In particular, we have

$$F^g(\beta) \circ \alpha^{(k)} = \alpha^{(k+h)} \circ F^g(\beta),$$

which proves that  $(\alpha^{(k)} \mid k \in G)$  is in  $\mathcal{Z}^G(\Lambda^{\text{op}})$  by Lemma 31.

That the  $G$ -grading is preserved follows by construction. Now, take an element in  $\mathcal{Z}^G(A)_{g,\chi}$ , for  $g \in G$  and  $\chi \in \hat{G}$ . By Corollary 34, this corresponds to a natural transformation  $\eta : \text{Id} \rightarrow \Pi_g$  satisfying  $\eta_{\Pi_k} = \chi(k) \Pi_k(\eta)$ , for all  $k \in G$ . Therefore

$$\alpha^{(k)} = F^g(\eta_{\Pi_k X_\bullet}) = \chi(k) F^g(\Pi_k(\eta_{X_\bullet})) = \chi(k) F^g(\eta_{X_\bullet}) = \chi(k) \alpha^{(0)}.$$

Under the isomorphism in Lemma 31 (or Lemma 24), the element  $(\alpha^{(k)} \mid k \in G) \in (\Lambda^{\text{op}})^{\oplus G}$  is thus mapped to  $(\alpha^{(0)}, \chi) \in \Lambda^{\text{op}}[\hat{G}]$ . By Definition 28(i), the original element is thus in the  $\chi$ -component of the  $\hat{G}$ -grading on  $\mathcal{Z}^G(\Lambda^{\text{op}})$ . This completes the proof.  $\square$

**Lemma 38.** Consider  $X_\bullet \in \mathcal{D}^g$ , such that

- for all  $g \in G$ ,  $\text{Hom}_{\mathcal{D}^g}(X_\bullet, \Pi_g X_\bullet[j]) = 0$ , unless  $j = 0$ ;
- $\text{add}(\oplus_{g \in G} \Pi_g X_\bullet)$  generates  $\mathcal{D}^g$  as a triangulated category;

then  $\Delta\zeta_{X_\bullet}$  is injective.

*Proof.* Since  $A\text{-gmod}$  is a  $\mathbb{k}$ -linear abelian category with finite dimensional homomorphism spaces and finitely many simple objects up to isomorphism, it is equivalent to  $B\text{-mod}$  for some finite dimensional algebra  $B$ . In fact, one can take

$$B := \text{end}_A(\Pi A).$$

Under this equivalence,  $\bigoplus_{g \in G} \Pi_g X_\bullet$  is mapped to a tilting complex in  $\mathcal{D}(B\text{-mod})$ . The conclusion thus follows from Lemma 21 and Definition 36.  $\square$

**5.3. The  $G$ -centre and Gradable derived equivalences.** Following [CoM, Section 3.2], we use the term “gradable derived equivalence” for an equivalence which commutes both with grading shifts and the suspension functor.

**Definition 39.** Consider two  $G$ -graded algebras  $A$  and  $B$ .

- (i) A functor  $H : \mathcal{D}^b(A\text{-gmod}) \rightarrow \mathcal{D}^b(B\text{-gmod})$  is “graded” if it intertwines the  $G$ -actions  $\Pi$ , as in 2.2.2.
- (ii) A gradable derived equivalence between two  $G$ -graded algebras  $A$  and  $B$  is a graded and triangulated functor  $F : \mathcal{D}^b(A\text{-gmod}) \rightarrow \mathcal{D}^b(B\text{-gmod})$  which admits an inverse which is also a graded and triangulated functor.

The following is a generalisation of [Ri1, Proposition 9.2] to  $G$ -graded algebras and an analogue of Theorem 18.

**Theorem 40.** *If two  $G$ -graded algebras  $A$  and  $B$  are gradable derived equivalent, then  $\mathcal{Z}^G(A) \cong \mathcal{Z}^G(B)$  as  $G \times \hat{G}$ -graded algebras.*

*Proof.* Let  $F : \mathcal{D}^b(A\text{-gmod}) \rightarrow \mathcal{D}^b(B\text{-gmod})$  denote a gradable derived equivalence. We will write  $\mathcal{D}^g$  for  $\mathcal{D}^b(B\text{-gmod})$ . We set  $X_\bullet \in \mathcal{D}^g$  equal to  $F(A)$ . By Lemma 7, we have a  $G$ -action preserving algebra isomorphism

$$\text{End}_{\mathcal{D}^g}(\Pi X_\bullet) \cong \text{end}_A(\Pi A).$$

By Lemma 25, we thus have

$$\text{End}_{\mathcal{D}^g}(\Pi X_\bullet)^G \cong A^{\text{op}},$$

as  $G$ -graded algebras. By Lemma 37, we then have a morphism of  $G \times \hat{G}$ -graded algebras

$$\Delta\zeta_{X_\bullet}^\Pi : \mathcal{Z}^G(B)^{\text{op}} \rightarrow \mathcal{Z}^G(A)^{\text{op}}.$$

This morphism is injective by Lemma 38.

By symmetry in the definition of gradable derived equivalences, the theorem now follows.  $\square$

## 6. SUPERALGEBRAS

We consider the special case  $G = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ , so we assume  $\text{char}(\mathbb{k}) \neq 2$ .  $G$ -graded algebras are then also known as superalgebras and the category  $A\text{-gmod}$  is known as the category of supermodules.

**6.1. Super, anti and ghost centre.** The character group is  $\hat{G} = \{\chi_0, \chi_1\}$ , where  $\chi_0(\bar{1}) = 1$  and  $\chi_1(\bar{1}) = -1$ . For the interpretation of  $G$ -graded algebras as superalgebras, some terminology appeared in [Go], which we generalise to arbitrary algebras and link with our constructions.

The *super centre* of  $A$ , denoted by  $s\mathcal{Z}(A)$ , is the subalgebra of  $A$  spanned by homogeneous elements  $x$  satisfying

$$(6.1) \quad xy = (-1)^{\partial x \partial y} yx,$$

for all homogeneous  $y \in A$ . The *anti centre*, denoted by  $a\mathcal{Z}(A)$ , is a subspace of  $A$  spanned by homogeneous elements  $x$  satisfying

$$(6.2) \quad xy = (-1)^{(\partial x + 1)\partial y} yx.$$

Generally, the anti centre does not constitute a subalgebra. The product of two elements of  $a\mathcal{Z}(A)$  belongs to  $s\mathcal{Z}(A)$ . The subalgebra of  $A$  consisting of linear combinations of elements of the super and the anti centre is known as the *ghost centre*,  $\tilde{\mathcal{Z}}(A) = s\mathcal{Z}(A) + a\mathcal{Z}(A)$ .

We can rewrite equation (6.1) as

$$xy = \begin{cases} y_{\chi_0} x, & \text{if } x \in A_{\bar{0}}; \\ y_{\chi_1} x, & \text{if } x \in A_{\bar{1}}. \end{cases}$$

Similarly, equation (6.2) becomes

$$xy = \begin{cases} y_{\chi_1} x, & \text{if } x \in A_{\bar{0}}; \\ y_{\chi_0} x, & \text{if } x \in A_{\bar{1}}. \end{cases}$$

This leads to the following interpretation of Definition 28(i).

**Proposition 41.** *For  $G = \mathbb{Z}_2$ , the  $G \times \hat{G}$ -grading of  $\mathcal{Z}^G(A)$  satisfies*

$$(i) \quad s\mathcal{Z}(A) = \mathcal{Z}^G(A)_{\bar{0}, \chi_0} \oplus \mathcal{Z}^G(A)_{\bar{1}, \chi_1};$$

$$(ii) \quad a\mathcal{Z}(A) = \mathcal{Z}^G(A)_{\bar{0}, \chi_1} \oplus \mathcal{Z}^G(A)_{\bar{1}, \chi_0}.$$

*As vector spaces, we hence have*

$$\mathcal{Z}^G(A) = s\mathcal{Z}(A) \oplus a\mathcal{Z}(A),$$

*where the latter direct sum is abstract, not inside  $A$ .*

Definition 28(ii) then yields the following.

**Proposition 42.** *For  $G = \mathbb{Z}_2$ , the ghost centre  $\tilde{\mathcal{Z}}(A)$  is equal to  $\underline{\mathcal{Z}}^G(A)$ . In particular, as subalgebras of  $A$ , we have*

$$\underline{\mathcal{Z}}^G(A) = s\mathcal{Z}(A) + a\mathcal{Z}(A).$$

We end this subsection with an example in which we demonstrate all the above notions for a small  $\mathbb{Z}_2$ -graded algebra.

**Example 43.** Consider the algebra  $A := \mathbb{k}[x]/(x^2)$  of dual numbers. We set  $A = \mathbb{k} \oplus \mathbb{k}x$  and consider  $A$  as a  $\mathbb{Z}_2$ -graded algebra with  $A_{\bar{0}} = \mathbb{k}$  and  $A_{\bar{1}} = \mathbb{k}x$ . We have  $\mathcal{Z}^G(A)_{\chi_0} = \mathcal{Z}(A) = A$  and  $\mathcal{Z}^G(A)_{\chi_1} = A_{\bar{1}}$ . Clearly  $\underline{\mathcal{Z}}^G(A) = A$  does not inherit the  $\hat{G}$ -grading. It follows that  $s\mathcal{Z}(A) = A$  and  $a\mathcal{Z}(A) = A_{\bar{1}}$ .

**6.2. Derived equivalences of superalgebras.** For a superalgebra  $A$ , we set  $\Pi_0 = \text{Id}$  as usual, and  $\Pi := \Pi_1$ . The category  $A\text{-gmod}$  is then a  $\Pi$ -category in the sense of [BE, Definition 1.6(i)].

Let  $A$  and  $B$  be superalgebras. According to [BE, Definition 1.6(ii)], a  $\Pi$ -functor in our setting is a functor  $F$  from  $A\text{-gmod}$  to  $B\text{-gmod}$ , or their derived categories, with a fixed natural isomorphism  $\xi^F : \Pi \circ F \rightarrow F \circ \Pi$  such that  $\xi_\Pi^F \circ \Pi(\xi^F)$  equals the identity natural transformation of  $F$ , when interpreted using  $\Pi^2 = \text{Id}$ . We thus conclude that  $F$  is a  $\Pi$ -functor if and only if  $F$  intertwines the  $\Pi$ -actions as in 2.2.2.

**Theorem 44.** *Let  $A, B$  be superalgebras and  $F$  a triangulated  $\Pi$ -equivalence which admits a triangulated  $\Pi$ -functor as inverse. Then we have algebra isomorphisms*

$$s\mathcal{Z}(A) \cong s\mathcal{Z}(B) \quad \text{and} \quad s\mathcal{Z}(A) \oplus a\mathcal{Z}(A) \cong s\mathcal{Z}(B) \oplus a\mathcal{Z}(A).$$

*Proof.* By Theorem 40, we have an equivalence of  $G \times \hat{G}$ -graded algebras  $\mathcal{Z}^G(A) \cong \mathcal{Z}^G(B)$ . The conclusions thus follow from Proposition 41.  $\square$

This implies that, under appropriate derived equivalences of superalgebras, the super centre is preserved, as well as the exterior sum of the super and the anti centre. Whether the ghost centre is also preserved does not follow from the general theory.

### 6.3. Alternative categorical realisations of the supercentre.

**6.3.1. Supernatural transformations.** For a  $\mathbb{Z}_2$ -graded algebra  $A$ , we introduce *the supercategory of modules*  $\mathcal{C} = A\text{-smod}$ . This  $\mathbb{k}$ -linear category has the same objects as  $A\text{-gmod}$ , but larger spaces of homomorphisms. For two graded modules  $M, N$ , the space of morphism  $\text{Hom}_{\mathcal{C}}(M, N)$  in  $A\text{-smod}$  is the  $\mathbb{Z}_2$ -graded vector space, with  $\text{Hom}_{\mathcal{C}}(M, N)_0 = \text{hom}_A(M, N)$  (the  $A$ -module morphism respecting the grading) and  $\text{Hom}_{\mathcal{C}}(M, N)_1$ , the elements  $f$  of

$$\text{Hom}_{\mathbb{k}}(M_0, N_1) \oplus \text{Hom}_{\mathbb{k}}(M_1, N_0) \subset \text{Hom}_{\mathbb{k}}(M, N),$$

which satisfy  $f(av) = (-1)^{\partial a} a f(v)$ , for homogenous  $a \in A$  and  $v \in M$ . The category  $A\text{-smod}$ , contrary to  $A\text{-mod}$  and  $A\text{-gmod}$ , will not be abelian in general. Already for  $A = A_0 = \mathbb{k}$ , kernels of epimorphisms in  $\mathbb{k}\text{-smod}$  are not always *graded* vector spaces.

We have

$$\text{End}_{A\text{-smod}}(A) \cong A^{\text{sop}},$$

with  $A^{\text{sop}}$  the superalgebra with underlying vector space  $A$  and multiplication given by

$$m(a, b) = (-1)^{\partial a \partial b} ba.$$

We, clearly, have

$$s\mathcal{Z}(A^{\text{sop}}) \cong s\mathcal{Z}(A) \cong s\mathcal{Z}(A)^{\text{sop}}.$$

Following, [BE, Definition (1.1)], a *supercategory*, resp. *superfunctor*, is a category, resp. functor, enriched over the category  $\mathbb{k}\text{-gmod}$  (here  $\mathbb{k}$  is  $\mathbb{Z}_2$ -graded as  $\mathbb{k} = \mathbb{k}_0$ ). The category  $A\text{-smod}$  is an example of a supercategory. We consider the category  $\mathcal{E}_A^s$  of superfunctors on  $A\text{-smod}$ . Morphism are given by *supernatural transformations*, see [BE, Definition (1.1)(iii)]. Concretely  $\text{Hom}_{\mathcal{E}_A^s}(F, G)_0$  is spanned by all natural transformations  $\eta$  such that  $\eta_M$  is even for each  $M \in A\text{-smod}$ . An element of  $\text{Hom}_{\mathcal{E}_A^s}(F, G)_1$  is a family of odd morphisms  $\{\eta_M, M \in A\text{-smod}\}$  in  $A\text{-smod}$  such that  $\eta_N \circ f = (-1)^{\partial f} f \circ \eta_M$ , for any  $f : M \rightarrow N$ .

**Proposition 45.** *We have an isomorphism of superalgebras*

$$\mathrm{End}_{\mathcal{E}_A^s}(\mathrm{Id}) \cong s\mathcal{Z}(A).$$

*Proof.* We consider the ordinary evaluation

$$\mathrm{End}_{\mathcal{E}_A^s}(\mathrm{Id}) \rightarrow \mathrm{End}_{A\text{-smod}}(A) \cong A^{\mathrm{sop}}.$$

Since, for any  $M$  in  $A\text{-smod}$  and  $v \in M$ , there exists  $\alpha \in \mathrm{Hom}_{A\text{-smod}}(A, M)$  with  $v \in \mathrm{Im}(\alpha)$ , this evaluation is injective.

A homogeneous supernatural transformation  $\eta : \mathrm{Id} \rightarrow \mathrm{Id}$  satisfies

$$\eta_A \circ \alpha = (-1)^{\partial \alpha \partial \eta} \alpha \circ \eta_A,$$

for each homogeneous morphism  $\alpha : A \rightarrow A$ . We set  $a := \eta_A(1)$ . The above equation then implies that  $a \in s\mathcal{Z}(A)$ . Every supernatural transformation thus yields an element of the supercentre.

Now we start from a homogenous  $a \in s\mathcal{Z}(A)$  and define, for each module  $M$ , morphisms  $\eta_M \in \mathrm{End}_{A\text{-smod}}(M)$  by

$$\eta_M(v) = av.$$

These form a supernatural transformation, completing the proof.  $\square$

**6.3.2.  $\Pi$ -natural transformations.** Recall the notion of  $\Pi$ -functors on  $A\text{-gmod}$  from Subsection 6.2. We follow the convention where  $\mathrm{Id}$  and  $\Pi$  are  $\Pi$ -functors where  $\xi^{\mathrm{Id}}$  is the identity and  $\xi^\Pi$  minus the identity. Following [BE, Definition 1.6(iii)], a  $\Pi$ -natural transformation between two  $\Pi$ -functors  $F$  and  $K$  on  $A\text{-gmod}$ , is a natural transformation  $\eta : F \rightarrow K$  such that

$$\eta_\Pi \circ \xi^F = \xi^K \circ \Pi(\eta),$$

inside  $\mathrm{Hom}_{\mathcal{E}_A^G}(\Pi \circ F, K \circ \Pi)$ . We let  $\mathrm{Hom}_{\mathcal{E}_A^G}^\Pi$  denote the spaces of  $\Pi$ -natural transformations. The subalgebra of  $\mathrm{End}_{\mathcal{E}_A^G}(\Pi)$  given by

$$\mathrm{End}_{\mathcal{E}_A^G}^\Pi(\Pi) = \mathrm{End}_{\mathcal{E}_A^G}^\Pi(\mathrm{Id}) \oplus \mathrm{Hom}_{\mathcal{E}_A^G}^\Pi(\mathrm{Id}, \Pi) \oplus \mathrm{Hom}_{\mathcal{E}_A^G}^\Pi(\Pi, \mathrm{Id}) \oplus \mathrm{End}_{\mathcal{E}_A^G}^\Pi(\Pi),$$

admits a natural free  $G$ -action.

**Proposition 46.** *We have an isomorphism of superalgebras*

$$\mathrm{End}_{\mathcal{E}_A^G}^\Pi(\Pi)^G \cong s\mathcal{Z}(A)^{\mathrm{op}}.$$

*Proof.* As an immediate consequence of Theorem 30 and Corollary 34, we have

$$\mathrm{End}_{\mathcal{E}_A^G}^\Pi(\Pi)^G = \mathrm{End}_{\mathcal{E}_A^G}(\Pi)_{0, \chi_0}^G \oplus \mathrm{End}_{\mathcal{E}_A^G}(\Pi)_{1, \chi_1}^G \cong \mathcal{Z}^G(A)_{0, \chi_0}^{\mathrm{op}} \oplus \mathcal{Z}^G(A)_{1, \chi_1}^{\mathrm{op}}.$$

The result then follows from Proposition 41(i).  $\square$

## 7. $G$ -HOCHSCHILD COHOMOLOGY SPECULATIONS

By Theorems 13 and 30, it is natural to introduce the following objects for an algebra  $A$  with an  $H$ -action, respectively a  $G$ -grading.

- $\mathrm{Ext}_{\mathcal{E}_A}^\bullet(\Phi)^H$ ;
- $\mathrm{Ext}_{\mathcal{E}_A^G}^\bullet(\Pi)^G$ .

These can be interpreted as generalisations of Hochschild cohomology, see e.g. [He, Chapter 7].

Based on Theorems 18 and 40 and [Ri2, Proposition 2.5], we arrive at the following natural questions:

- (1) Consider a  $G$ -graded algebra  $A$  with the associated  $\hat{G}$ -action  $\phi$ . Do we have  $\text{Ext}_{\mathcal{E}_A}^\bullet(\Phi)^{\hat{G}} \cong \text{Ext}_{\mathcal{E}_A}^\bullet(\Pi)^G$ ?
- (2) For two algebras  $A$  and  $B$  with  $H$ -actions  $\phi$  and  $\omega$  and an equivalence of triangulated categories  $\mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B)$  intertwining  $\Phi$  and  $\Omega$ , do we have  $\text{Ext}_{\mathcal{E}_A}^\bullet(\Phi)^H \cong \text{Ext}_{\mathcal{E}_B}^\bullet(\Omega)^H$ ?
- (3) If two  $G$ -graded algebras  $A$  and  $B$  are gradable derived equivalent, do we have  $\text{Ext}_{\mathcal{E}_A}^\bullet(\Pi)^G \cong \text{Ext}_{\mathcal{E}_B}^\bullet(\Pi)^G$ ?

## APPENDIX A. PROOFS OF SECTION 2

*Proof of Proposition 4.* To prove this, consider the diagram given in Figure 1. All edges of this diagram correspond to the obvious pair of mutually inverse isomorphisms (given by using horizontal pre- and post-composition of  $\alpha$ ,  $\beta$  or  $\xi$  with necessary identity morphisms). Note that the vertical edge in the middle of the diagram is induced from either  $\alpha$  or  $\beta$ , where equality of both options follows from the counit-unit adjunction formula  $K(\alpha) \circ \beta_K = 1_K$ .

The bottom triangle commutes because of commutativity of (2.3). To check commutativity of all rectangles one uses associativity of horizontal composition and interchange law. This implies that the whole diagram commutes and establishes our claim.  $\square$

*Proof of Proposition 5.* Let  $\alpha$  denote an isomorphism of functors  $F \circ F^{-1} \rightarrow \text{Id}_{\mathcal{D}}$ . Using the notation of 2.2.2, we have isomorphisms of functors

$$\delta_k = \Upsilon_k(\alpha) \circ \xi_{F^{-1}}^k : \mathbf{F}(\Gamma_k) = F \circ \Gamma_k \circ F^{-1} \rightarrow \Upsilon_k.$$

We have the corresponding isomorphism of algebras  $\beta : \text{End}_{\text{Func}\mathcal{C}}(\Gamma) \rightarrow \text{End}_{\text{Func}\mathcal{D}}(\Upsilon)$ , which maps  $\eta \in \text{Hom}(\Gamma_k, \Gamma_h)$  to

$$\beta(\eta) = \delta_h \circ \mathbf{F}(\eta) \circ \delta_k^{-1}.$$

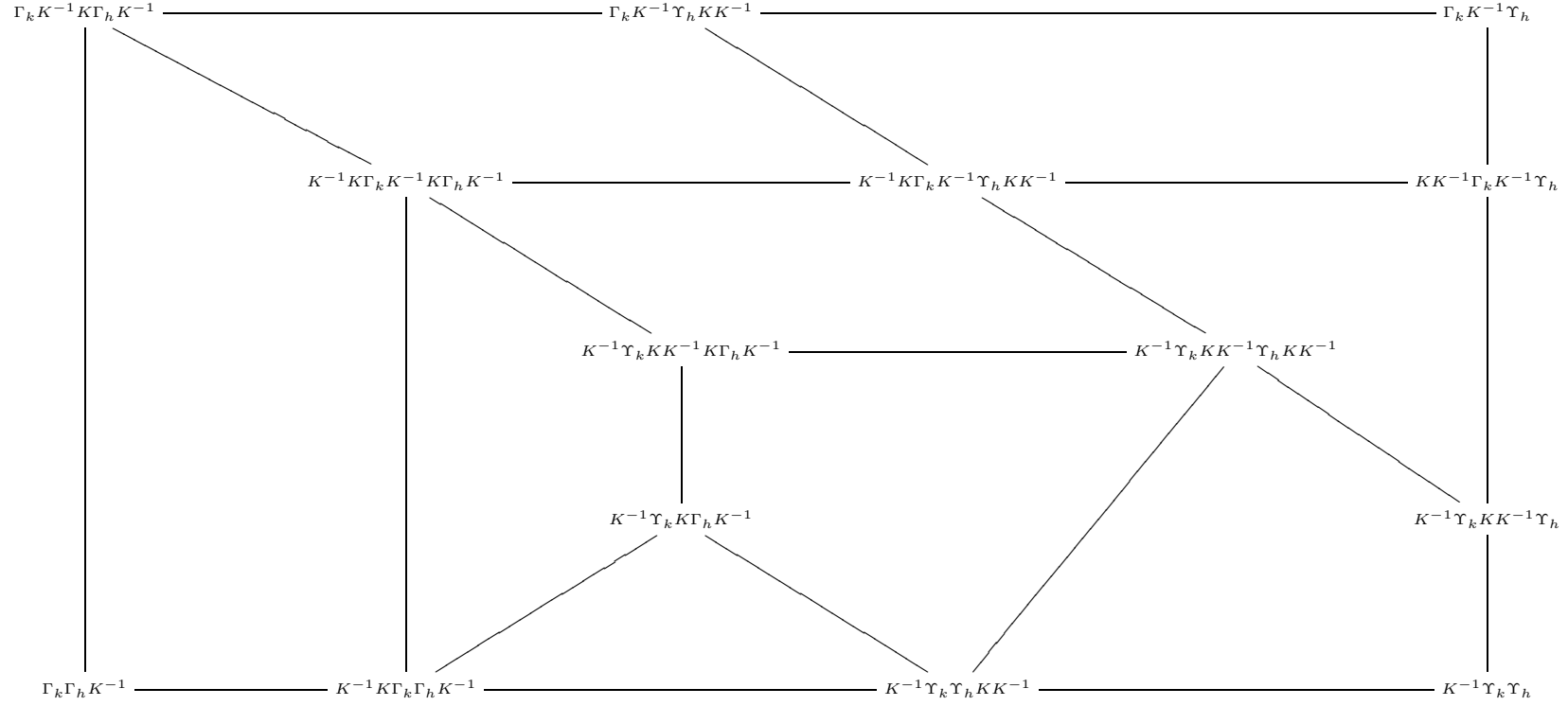
The claim is now that  $\beta$  intertwines the  $H$ -actions, meaning that

$$\beta(\Gamma_g(\eta)) = \Upsilon_g(\beta(\eta)),$$

for all  $g \in H$ . The left-hand side of the above equation is calculated, using equation (2.2) and the definition of  $\xi^g$ , to be

$$\begin{aligned} \delta_{gh} \circ \mathbf{F}(\Gamma_g(\eta)) \circ \delta_{gk}^{-1} &= \Upsilon_{gh}(\alpha) \circ \xi_{F^{-1}}^{gh} \circ (F \circ \Gamma_g)(\eta_{F^{-1}}) \circ \delta_{gk}^{-1} \\ &= \Upsilon_{gh}(\alpha) \circ \Upsilon_g(\xi_{F^{-1}}^h) \circ \xi_{\Gamma_h \circ F^{-1}}^g \circ (F \circ \Gamma_g)(\eta_{F^{-1}}) \circ \delta_{gk}^{-1} \\ &= \Upsilon_{gh}(\alpha) \circ \Upsilon_g(\xi_{F^{-1}}^h) \circ (\Upsilon_g \circ F)(\eta_{F^{-1}}) \circ \xi_{\Gamma_k \circ F^{-1}}^g \circ \delta_{gk}^{-1} \\ &= \Upsilon_{gh}(\alpha) \circ \Upsilon_g(\xi_{F^{-1}}^h) \circ (\Upsilon_g \circ F)(\eta_{F^{-1}}) \circ (\Upsilon_{gk}(\alpha) \circ \Upsilon_g(\xi_{F^{-1}}^k))^{-1} \\ &= \Upsilon_g(\Upsilon_h(\alpha) \circ \xi_{F^{-1}}^h \circ F(\eta_{F^{-1}}) \circ (\Upsilon_k(\alpha) \circ \xi_{F^{-1}}^k)^{-1}). \end{aligned}$$

By definition, this is  $\Upsilon_g(\beta(\eta))$ , which concludes the proof.  $\square$



THE  $G$ -CENTRE

FIGURE 1. Commutative diagram in the proof of Proposition 4

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